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SOME GAME PROBLEMS IN DISTRIBUTED CONTROLLED SYSTEMS†

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Differential games of pursuit and evasion are considered in a system described by a partial differential equation containing an elliptic operator and additively occurring control parameters. Spaces are introduced using generalized eigenvalues and generalized eigenfunctions of this operator which depend on a non-negative parameter. Four versions of the formulation of the game problems are studied which differ in the constraints imposed on the control of the players. In the case of two of the versions, sufficient conditions are presented such that, when these are satisfied, evasion is possible from all initial states (the pursuit problem for these games has been studied earlier). In the case of the third version, two infinite non-intersecting sets are distinguished such that the completion of a pursuit is possible from the points of one of them and evasion is possible from the points of the second set. In the case of the fourth version, the possibility of completing a pursuit from any initial position in an arbitrary small neighbourhood of zero is demonstrated.

Some of the results described below were presented in [1]. Arguments which were employed in the finite-dimensional case in [2] are used to solve the invasion problem.

1. INTRODUCTION

A differential operator A of the form [3]

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right), \quad x \in \Omega, \quad a_{ij}(x) = a_{ji}(x) \in C^1(\bar{\Omega}) \tag{1.1}$$

is considered in the space $L_2(\Omega)$, where Ω is a domain in R^n bounded by a piecewise-smooth boundary, $n \geq 1$ and $\bar{\Omega}$ is its closure. $\mathring{C}^2(\Omega)$ (the space of doubly continuous differentiable finite functions) is the domain of definition $D(A)$ of the operator A. The coefficients $a_{ij}(x)$ satisfy the following conditions: a constant $\gamma > 0$ exists such that, for all $x \in \Omega$ and $(\xi_1, \xi_2, \dots, \xi_n) \in R^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} \xi_i^2 \tag{1.2}$$

It can be shown that the operation

$$(z, y)_A = (Az, y), \quad z, y \in \mathring{C}^2(\Omega)$$

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satisfies all of the conditions of a scalar product [4], Hence, $\mathring{C}^2(\Omega)$ turns into a partial Hilbert space. On completing the space using the norm

$$\|z\|_A = (Az, z)^{1/2}, \quad z \in \mathring{C}^2(\Omega)$$

we obtain the complete Hilbert space, which is called the energy space of the operator A: we will denote this space by H_A.

It is well known that the operator A (1.1) has a discrete spectrum or, more accurately, it has an infinite sequence $0 < \lambda_1 \le \lambda_2 \le \ldots$ of generalized eigenvalues with the limit at infinity and a sequence of generalized eigenfunctions $\varphi_1, \varphi_2 \ldots$ which is complete in $L_2(\Omega)$ and H_A. We shall assume that $(\varphi_i, \varphi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Suppose r is an arbitrary non-negative number. We introduce the following spaces (henceforth summation is carried out from $i = 1$ to $i = \infty$)

$$l_r = \{\alpha = (\alpha_1, \alpha_2, \ldots) : \sum \lambda_i^r \alpha_i^2 < \infty\}, \quad H_r(\Omega) = \{f \in L_2(\Omega) : f = \sum \alpha_i \varphi_i, \alpha \in l_r\} \tag{1.3}$$

with the scalar products and norms

$$\begin{aligned} &(\alpha, \beta)_r = \sum \lambda_i^r \alpha_i \beta_i, \quad \alpha, \beta \in l_r, \quad \|\alpha\| = \|f\| = (\alpha, \alpha)_r^{1/2} \\ &(f, g)_r = (\alpha, \beta)_r, \quad g = \sum \beta_i \varphi_i \end{aligned} \tag{1.4}$$

Note that $H_0(\Omega) = L_2(\Omega)$ and $H_r(\Omega) \subset H_s(\Omega)$ for arbitrary $s, r, 0 \le s \le r$.

We denote by $(0, T; H_r(\Omega))$ $(L_2(0, T; H_r(\Omega)))$ the space of continuous (square integrable, measurable) functions defined in the interval $[0, T]$ with values in $H_r(\Omega)$, where T is a certain positive constant.

2. DETERMINATION OF THE POSSIBILITY OF EVASION AND THE POSSIBILITY OF COMPLETING A PURSUIT

We will consider the following conflict controlled distributed system (distributed differential game)

$$\begin{aligned} &\frac{dz(t)}{dt} + Az(t) = -u(t) + \upsilon(t), \quad 0 < t \le T \\ &u(\cdot), \quad \upsilon(\cdot) \in L_2(0, T; H_r(\Omega)), \quad z(0) = z^{(0)}, \quad z^{(0)} \in H_{r+1}(\Omega) \end{aligned} \tag{2.1}$$

The controls $u(\cdot)$ and $\upsilon(\cdot)$ of the first (pursuing) and the second (pursued) players are respectively assumed to satisfy one of the following systems of inequalities

$$\begin{aligned} &1) \quad \|u(t)\| \le \rho, \quad \|\upsilon(t)\| \le \sigma, \quad 0 \le t \le T \\ &2) \quad \|u(\cdot)\| \le \rho, \quad \|\upsilon(\cdot)\| \le \sigma \\ &3) \quad \|u(\cdot)\| \le \rho, \quad \|\upsilon(t)\| \le \sigma, \quad 0 \le t \le T \\ &4) \quad \|u(t)\| \le \rho, \quad 0 \le t \le T, \quad \|\upsilon(\cdot)\| \le \sigma \end{aligned} \tag{2.2}$$

where ρ and σ are non-negative constants.

We shall subsequently call system (2.1), where the functions $u(\cdot)$ and $\upsilon(\cdot)$ satisfy one of the systems of inequalities *1, 2, 3* or *4*, Game *1, 2, 3* or *4* respectively. We shall call the point $z^{(0)}$ the initial position of the point z (or of the games).

We now formulate the determination of the possibility of evasion and the possibility of completing a pursuit from an initial position $z^{(0)}$ (henceforth $z^{(0)} \ne 0$).

Definition 1. In Game 1 (Game 2 and 3) evasion is possible from the initial position $z^{(0)}$, if, using any number $T > 0$ and an arbitrary control $u(t)$, $0 \le t \le T$ which satisfies the condition $\|u(t)\| \le \rho(\|u(\cdot)\| \le \rho)$, it is possible to construct a control $\upsilon(t)$, $0 \le t \le T$, such that the solution $z(t)$, $0 \le t \le T$ of problem (2.1) does not vanish. In addition: 1) to find the vector of $\upsilon(t)$, it is permitted to use $z^{(0)}$ ($z^{(0)}$ and $z^{(0)}$, $u(s)$, $t - \theta \le s < t$, where $\theta > 0$ is an arbitrary constant and $z^{(0)}$, $u(s)$, $0 \le s < t$, if $0 \le t < \theta$; 2) the function $\upsilon(t)$, $0 \le t \le T$ satisfied the inequality $\|\upsilon(t)\| \le \sigma(\|\upsilon(\cdot)\| \le \sigma; \|\upsilon(t)\| \le \sigma)$.

Definition 2. In Game 3 (Game 4) completion of a pursuit from an initial position $z^{(0)}$ is possible if a number $T = T(z^{(0)})$ and a function $u(\upsilon, t)$, $\upsilon \in R^1$, $0 \le t \le T$ exist such that, for an arbitrary control $\upsilon(t)$, $0 \le t \le T$ which satisfies the inequality $\|\upsilon(t)\| \le \sigma(\|\upsilon(\cdot)\| \le \sigma)$, the solution $z(t)$, $0 \le t \le T$ of problem (2.1) vanishes for a certain $t = t' \in [0, T]$ (finds itself in an arbitrary ϵ-neighbourhood of zero, that is, $\|z(t)\| \le \epsilon$ for a certain $t = t' \in [0, T]$). In addition, the function $u(t) = u(\upsilon(t), t)$ satisfies the inequality $\|u(\cdot)\| \le \rho(\|u(t)\| \le \rho)$.

The pursuit problem has been studied earlier in [5, 6] and, in particular, when $\rho > \sigma$, the possibility of completing a pursuit in Game 2 from an arbitrary initial position $z^{(0)}$ has been established [5]. Interesting classes of control problems are considered in [3, 7–10].

3. MAIN RESULTS

Theorem 1. If $\sigma \ge \rho$, then evasion is possible from an arbitrary initial position $z^{(0)}(\neq 0)$ in Game 1 and 2.

Theorem 2. 1. In the case of Game 3, two infinite non-intersecting sets of initial positions exist for arbitrary $\rho > 0$ and $\sigma \ge 0$ such that completion of a pursuit is possible from the points of the first of them and evasion is possible from the points of the second of them.

2. In Game 4, completion of a pursuit is possible from an arbitrary initial position $z^{(0)}$ for arbitrary $\rho > 0$ and $\sigma \ge 0$.

Proof of Theorem 1. Suppose $z^{(0)}$ is an arbitrary initial position, T is an arbitrary positive number, and $u(\cdot)$ and $\upsilon(\cdot)$ are controls which satisfy condition *1* (condition *2*). We substitute these controls into the right-hand side of Eq. (2.1) and, in order to find an explicit form of the solution of the resulting problem $z(t)$, $0 \le t \le T$, we substitute the Fourier series

$$(u(t), v(t), z(t), z^0) = \sum (u_i(t), v_i(t), z_i(t), z_i^0)\varphi_i, \quad u_i(\cdot), v_i(\cdot), z_i(\cdot) \in L_2[0, T]$$

into system (2.1), where

$$\|F(t)\|^2 = \sum \lambda_i^r F_i^2(t) \quad \left(\|F(\cdot)\|^2 = \int_0^T \|F(t)\|^2 dt\right), \quad F = u, v$$

and equate the corresponding Fourier coefficients. As a result, we obtain the infinite system of differential equations and initial conditions

$$\begin{aligned} &\frac{dz_i(t)}{dt} = -\lambda_i z_i(t) - w_i(t), \quad 0 < t \le T, \quad z_i(0) = z_i^{(0)}, \quad i = 1, 2, \dots; \\ &w_i(t) = u_i(t) - v_i(t) \end{aligned} \tag{3.1}$$

It is obvious that the functions

$$z_i(t) = e^{-\lambda_i t}\left[z_i^{(0)} - \int_0^t w^{\lambda_i s} w_i(s) ds\right], \quad 0 \le t \le T, \quad i = 1, 2, \dots \tag{3.2}$$

are a solution of system (3.1). It can be shown by direct calculations that the function

$$z(t) = \sum z_i(t)\varphi_i, \quad 0 \le t \le T$$

belongs to the space $C(0, T; H_{r+1}(\Omega))$ and is a solution of problem (2.1) in the sense of the theory of generalized functions [4].

Next, since $z^{(0)} \neq 0$ according to the condition, then $z_i(0) \neq 0$ for any $i = k$. Suppose, to be specific, that $z_k^{(0)} > 0$ (the case when $z_k^{(0)} < 0$ is treated in a similar way). Moreover, we assume that the inequalities *1* are satisfied (Game 2 is considered next).

For all $t \in [0, T]$, we put $\upsilon_i(t) = 0, i \neq k$ and $\upsilon_k(t) = \sigma\lambda_k^{r/2}$ (it is clear that $\|\upsilon(t)\| \leq \sigma$ in $[0, T]$). Since $\|u(t)\|^2 \leq \rho^2$, then, obviously, $|u_k(t)| \leq \rho\lambda_k^{-r/2}$. Hence,

$$\upsilon_k(t) - u_k(t) \geq \sigma\lambda_k^{-r/2} - \rho\lambda_k^{-r/2} \geq 0$$

(We recall that $\sigma \geq \rho$.)

Consequently (see expression (3.2)), for all $t \in [0, T]$

$$z_k(t) \geq e^{-\lambda_k t} z_k^{(0)} \tag{3.3}$$

whence it follows that $z_k(t) \neq 0$ in $[0, T]$ or $z_k^{(0)} > 0$. This means that $z(t) \neq 0$ in $[0, T]$ since, otherwise, a $t' \in [0, T]$ exists for which $z(t') = 0, z_k(t') = 0$.

Hence, evasion is possible in Game *1* from any initial position $z^{(0)}$.

Now, suppose the inequalities *2* are satisfied. As above, we choose a non-zero Fourier coefficient $z_k^{(0)}$ in the expression $z^{(0)}$ and assume that $z_k^{(0)} > 0$.

It is clear that a number $\delta \in (0, T]$ exists such that, if, in expression (3.2), $\upsilon_k(t) = 0$ in $[0, \delta]$ and $u_k(t)$, $0 \leq t \leq T$ is a square integrable function which satisfies the inequality

$$\int_0^T u_k^2(t)dt \leq \frac{\rho^2}{\lambda_k^r}$$

then $z_k(t) \neq 0$ in $[0, \delta]$. Moreover, satisfaction of the inequality $z_k(t) > z_k^{(0)}/2$ in $[0, \delta]$ can also be achieved.

We put $\upsilon_i(t) = 0$ for all $i \neq k$ and $t \in (0, T]$, and $\upsilon_k(t) = u_k(t - \delta)$ for all $t \in [\delta, T]$.

On taking account of these facts and decreasing δ, if it is necessary, it is possible also to achieve satisfaction of the inequality $z_k(t) \geq z_k^{(0)}/2$ in the interval $[\delta, T]$.

This means that, in the method proposed above, the inequality

$$z_k(t) \geq e^{-\lambda_k t} z_k^{(0)}/2, \quad 0 \leq t \leq T \tag{3.4}$$

holds for all $t \in [0, T]$ from which it follows that, in Game 2, evasion is possible from any initial position $z^{(0)}$.

Proof of Theorem 2. 1. Suppose T is an arbitrary positive number and k is an arbitrary nature number. We will denote by $X_k(T_0)$ the set of initial positions of the form $z^{(0)} = z_k^{(0)}\varphi_k$, where the coefficient $z_k^{(0)}$ and the number T_0 satisfy the conditions

$$0 < z_k^{(0)^2} < \frac{(\rho - \sigma\sqrt{T_0})^2 (e^{\lambda_k T_0} - 1)^2}{T_0 \lambda_k^{r+2}}, \quad 0 < T_0 \leq T, \quad \rho - \sigma\sqrt{T_0} > 0 \tag{3.5}$$

We will now that, in Game 3, a pursuit can be completed from an arbitrary initial position which satisfies the condition

$$z^{(0)} \in X = \bigcup_{T_0} \bigcup_{k=1}^{\infty} X_k(T_0) \tag{3.6}$$

Indeed, suppose $\upsilon(t)$, $0 \leq t \leq T$ is an arbitrary control of the second player, $\|\upsilon(t)\| \leq \sigma$. It is clear that

$$\|\upsilon(\cdot)\|_{T_0}^2 \equiv \int_0^{T_0} \|\upsilon(t)\|^2 dt \leq \sigma^2 T_0$$

Therefore, if we put $\upsilon(t) = u(t) + w(t)$ in $[0, T_0]$ and assume that $\|w(\cdot)\| \leq \rho - \sigma\sqrt{T_0}$, then it is clear that $\|u(\cdot)\|_{T_0} \leq \rho$.

Taking account of this remark, we now consider system (3.1). Since $z^{(0)} \in X$, then $z_i^{(0)} = 0$ for all $i \neq k$, and $z_k^{(0)}$ satisfies conditions (3.5).

We put

$$w_i(t) \equiv 0 \quad \text{for all} \quad i \neq k, \quad w_k(t) = \frac{\lambda_k z_k^{(0)}}{e^{\lambda_k T_0} - 1}, \quad 0 \le t \le T_0 \tag{3.7}$$

Since $w_i(t) = z_i^{(0)} = 0$ for all $i \neq k$, then, according to system (3.1), $z_i(t) \equiv 0$ in $[0, T_0]$ for each $i \neq k$. When $i = k$, from equality (3.2) we obtain: $z_k(T_0) = 0$. This means that, with the above-mentioned method of choosing the functions $u(\cdot)$ and $w(\cdot)$, Game 3 is completed from an arbitrary initial position, which satisfies condition (3.6), after a time $T(z^{(0)}) \le T$. It can be shown that $\|w(\cdot)\| \le \rho - \sigma\sqrt{T_0}$.

We will now show that, in Game 3, evasion is possible from any initial position which satisfies the condition

$$z^{(0)} \in Y = \bigcup_{k=1}^{\infty} Y_k, \quad Y_k = \left\{ z^{(0)} : z_k^{(0)^2} > \rho^2 \frac{e^{2\lambda_k T} - 1}{2\lambda_k} \right\} \tag{3.8}$$

In fact, it follows from condition (3.8) that $z^{(0)} \in Y_i$ for any $i = k$. Next, suppose $u(t)$, $0 \le t \le T$ is an arbitrary control and $\|u(\cdot)\| \le \rho$. We select the function $\upsilon(t)$, $0 \le t \le T$, which guarantees the possibility of evasion from a position $z^{(0)}$ in the following manner

$$\upsilon_i(t) \equiv 0, \quad i \neq k; \quad \upsilon_k(t) \equiv \sigma/\lambda_k^{r/2}$$

When $i = k$, the solution of problem (3.1) then has the form

$$z_k(t) = e^{\lambda_k t}\left[z_k^{(0)} + \xi_k(t) - \int_0^t e^{\lambda_k s} u_k(s) ds \right], \quad 0 \le t \le T, \quad \xi_k(t) = \frac{\sigma(e^{\lambda_k t} - 1)}{\lambda_k^{1+r/2}} \tag{3.9}$$

To be specific, suppose $z_k^{(0)} > 0$. Then, by virtue of the Cauchy–Bunyakovskii inequality

$$z_k^{(0)} + \xi_k(t) - \int_0^t e^{\lambda_k s} |u_k(s)| ds \ge z_k^{(0)} + \xi_k(t) - \rho\sqrt{\frac{e^{2\lambda_k T} - 1}{2\lambda_k}}$$

From this and from relations (3.8) and (4.9) for all $t \in [0, T]$, we obtain the inequality

$$z_k(t) > \frac{\sigma(1 - e^{-\lambda_k t})}{\lambda_k^{1+r/2}} \tag{3.10}$$

which means that, in Game *3*, evasion is possible from an arbitrary initial position which satisfies condition (3.8). It is clear that $\|\upsilon(t)\| \le \sigma$.

It is obvious that X and Y are infinite sets. Moreover, they do not intersect. Otherwise, it would be simultaneously possible to complete a pursuit and to evade an encounter from a certain initial position $z^0 \in X \cap Y$ which leads to a contradiction. The first part of the theorem is proved.

2. Suppose ϵ is an arbitrary positive number, $z^{(0)}$ is an arbitrary initial position, $\|z^{(0)}\| > \epsilon$, $\upsilon(t)$ is an arbitrary control of the second player and $\|\upsilon(\cdot)\| \le \sigma$.

We select the function $u(t)$ as follows (compare with expressions (3.7)):

$$u_i(t) = \frac{\lambda_i z_i^{(0)}}{e^{\lambda_i T_1} - 1}, \quad 0 \le t \le T_1, \quad i = 1, 2, \ldots, \quad T_1 = T(z^{(0)}) = \frac{\|z^{(0)}\|}{\rho}$$

Then, for arbitrary i, the solution of problem (3.1) has the form

$$z_i(t) = e^{-\lambda_i t}[z_i^{(0)} - I(u_i; t) + I(\upsilon_i; t)], \quad I(F_i; t) = \int_0^t e^{\lambda_i s} F_i(s) ds \tag{3.11}$$

By virtue of the choice of the function $u_i(t)$, $0 \le t \le T_1$, we obtain that $z_i^{(0)} - I(u_i; T_1) = 0$. Consequently, $z_i(T_1) = e^{-\lambda_i T_1} I(\upsilon_i; T_1)$.

Two cases are possible:

(1) $\|z(T_1)\| \le \epsilon$, (2) $\|z(T_1)\| > \epsilon$.

In case *1*, Game *4* is completed from an initial position z^0 at the instant of time T_1.

In case *2*, putting $z^0 = z(T_1)$, we repeat the preceding arguments and obtain

$$z_i(T_1 + T_2) = e^{-\lambda_i T_2} \int_0^{T_2} e^{\lambda_i s} \upsilon_i(T_1 + s)ds, \quad T_2 = T(z(T_1)) = \frac{\|z(T_1)\|}{\rho}$$

Here, two sub-cases are possible:

(*2a*) $\|z(T_1 + T_2)\| \le \epsilon$, (*2b*) $\|z(T_1 + T_2)\| > \epsilon$. In sub-case *2a*, Game *4* is completed at the instant of time $T_1 + T_2$. In sub-case *2b*, putting $z^{(0)} = z(T_1 + T_2)$, we use the earlier arguments, and so forth.

By arguing against this, it can be show that, up to the $(k + 1)$th step, where

$$k = [\sigma^2/(2\epsilon^2)] + 1$$

Game *4* will be completed from an initial position $z^{(0)}$ which guarantees a time of completion of the pursuit equal to $\rho^{-1}(\|z^{(0)}\| + \sigma^2/(2\epsilon))$.

Note that, considerable use is made of the boundedness of the energy of the evader when proving the theorem (see formula (2.2)).

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